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Imitation dynamics with time delay

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ABSTRACT

Based on the classic imitation dynamics (Hofbauer and Sigmund, 1998, *Evolutionary Games and Population Dynamics*, Cambridge University Press), the imitation dynamics with time delay is investigated, where the probability that an individual will imitate its opponent's own strategy is assumed to depend on the comparison between the past expected payoff of this individual's own strategy and the past expected payoff of its opponent's own strategy, i.e. there is a time delay effect. For the two-phenotype model, we show that if the system has an interior equilibrium and this interior equilibrium is stable when there is no time delay, then there must be a critical value of time delay such that the system tends to a stable periodic solution when the time delay is larger than the critical value. On the other hand, for three-phenotype (rock-scissors-paper) model, the numerical analysis shows that for the stable periodic solution induced by the time delay, the amplitude and the period will increase with the increase of the time delay. These results should help to understand the evolution of behavior based on the imitation dynamics with time delay.

1. Introduction

In the evolutionary game theory, the imitation dynamics provides a fundamental theoretical framework to investigate how the spreading of successful strategies is more likely to occur through imitation than through inheritance (Hofbauer and Sigmund, 1998; Weibull, 1995; Schlag, 1994, 1998; Vilone et al., 2012). Similar to the standard evolutionary game dynamics, the classic imitation dynamics also assume that the probability that an individual will imitate its opponent's own strategy depends instantaneously on the system state (i.e. the frequencies of different phenotypes in the population), or an individual's strategy not only depends on the current payoff of its own strategy but also instantaneously on the current payoff of its opponent's own strategy (Hofbauer and Sigmund, 1998). However, in real system it should be very difficult to imitate the opponent's own strategy in a moment according to the comparison between the current payoff of an individual's own strategy and the current payoff of its opponent's own strategy. So, a more reasonable assumption should be that an individual is able to imitate its opponent's own strategy with a certain probability according to the comparison between the past payoff of its own strategy and the past payoff of its opponent's own strategy. This assumption strongly implies that the effect of time delay should be considered in the imitation dynamics.

In fact, the replicator dynamics with time delay has been considered by some authors (Tao and Wang, 1997; Alboszta and Mie-kisz, 2004; Wesson and Rand, 2016). Their results show clear that the time delay will be able to change the dynamic properties of the evolutionary game, or the effect of time delay may be also an important factor for us to understand the evolution of behavior. In the previous studies, the key assumption is that the fitness of an individual depends on its expected payoff at a given past time. For the imitation dynamics with time delay, we are more interested in how the probability that an individual will imitate his opponent's strategy depends on the relative size of difference between their expected payoffs at a given past time. In this paper, the imitation dynamics with time delay is investigated. Our main is to show what will happen in an imitation dynamics when the effect of time delay is considered. The paper is organized as: the stability of a two-phenotype imitation dynamics with time delay is first analytically investigated in Subsection 2.1, where we define that the probability that an individual will imitate its opponent's own strategy (or will not change its own strategy) is proportional to the past payoff of its opponent's own strategy (or its own strategy); in Subsection 2.2 a three-phenotype imitation dynamics is considered, where, as an example, only the rock-scissors-paper game is investigated using numerical analysis; and in Section 3 we summarize our results and compare with the related previous studies dealing with replicator dynamics with time delay.

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2. Models and analysis

2.1. Two-phenotype imitation dynamics with time delay

We first consider a symmetric two-phenotype model with pure strategies S_1 and S_2, and with payoff matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, where a_{ij} is the payoff of an individual using phenotype S_i when it plays against an individual using phenotype S_j for $i, j = 1, 2$. For convenience and without loss of generality, we assume $a_{ij} \geq 0$ for $i, j = 1, 2$. Let x denote the frequency of phenotype S_1. Then, the expected payoff of phenotype S_i, denoted by f_i, can be given by $f_i = xa_{i1} + (1 - x)a_{i2}$ for $i = 1, 2$ (Maynard Smith, 1982; Hofbauer and Sigmund, 1998).

Basically, the classic imitation dynamics (Hofbauer and Sigmund, 1998) assumes that occasionally an individual is picked out of the population and afforded the opportunity to change his strategy. He samples another individual at random and adopts (or imitates) his strategy with a certain probability. Specifically, when an individual using phenotype S_i plays against an individual using phenotype S_j, the probability (or the imitation rate) that the S_j-strategist switches to S_i is denoted by F_{ij} for $i = 1, 2$. Furthermore, it is assumed that the imitation rate F_{ij} depends on the current expected payoffs $f_i(x)$ and $f_j(x)$:

$$F_{ij}(x) = F(f_i(x), f_j(x)) \tag{1}$$

for $i, j = 1, 2$, where the function $F(u, v)$ defines the imitation rule (the same for all individuals). Then, the time evolution of x can be easily given by

$$\begin{aligned} \tfrac{dx}{dt} &= x(1 - x)(F_{12}(x) - F_{21}(x)) \\ &= x(1 - x)(F(f_1(x), f_2(x)) - F(f_2(x), f_1(x))) \end{aligned} \tag{2}$$

(Hofbauer and Sigmund, 1998).

In this study, based on the rule "*individual is more likely to imitate the better*", we take $F(u, v) = u/(u + v)$, i.e. we define $F_{12} = f_1/(f_1 + f_2)$ and $F_{21} = f_2/(f_1 + f_2)$ (i.e. F_{12} is proportional to f_1 and F_{21} proportional to f_2). On the other hand, for our main goal, we also assume that the imitation rate (F_{12}, F_{21}) at time t depends on the expected payoffs at time $t - \tau$ (where τ denotes the time delay), i.e. we take

$$\begin{aligned} F_{12}(x(t - \tau)) &= \tfrac{f_1(x(t-\tau))}{f_1(x(t-\tau)) + f_2(x(t-\tau))}, \\ F_{21}(x(t - \tau)) &= \tfrac{f_2(x(t-\tau))}{f_1(x(t-\tau)) + f_2(x(t-\tau))}. \end{aligned} \tag{3}$$

Thus, Eq. (2) can be rewritten as a time-delay differential equation, or an imitation dynamics with time-delay

$$\begin{aligned} \tfrac{dx(t)}{dt} &= x(t)(1 - x(t))(F_{12}(x(t - \tau)) - F_{21}(x(t - \tau))) \\ &= x(t)(1 - x(t))\tfrac{f_1(x(t-\tau)) - f_2(x(t-\tau))}{f_1(x(t-\tau)) + f_2(x(t-\tau))}. \end{aligned} \tag{4}$$

For Eq. (4), it is easy to see that an interior equilibrium exists if and only if $a_{11} > a_{21}$ and $a_{22} > a_{12}$, or $a_{11} < a_{21}$ and $a_{22} < a_{12}$, which is given by

$$x^* = \frac{a_{12} - a_{22}}{a_{12} - a_{22} + a_{21} - a_{11}}. \tag{5}$$

Obviously, if $(a_{12} - a_{22})/(a_{12} - a_{22} + a_{21} - a_{11}) \notin (0, 1)$ (i.e. no interior equilibrium can exist), then the boundary $x = 0$ must be globally asymptotically stable if $a_{11} < a_{21}$ and $a_{12} < a_{22}$, or the boundary $x = 1$ is globally asymptotically if $a_{11} > a_{21}$ and $a_{12} > a_{22}$. For the situation with $x^* \in (0, 1)$(i.e., the interior equilibrium exists), Eq. (4) can be re-expressed as

$$\frac{dx(t)}{dt} = \frac{A_1}{A_2}\cdot\frac{x(t - \tau) - x^*}{x(t - \tau) + A_3}x(t)(1 - x(t)), \tag{6}$$

where $A_1 = a_{11} - a_{12} - a_{21} + a_{22}$, $A_2 = a_{11} - a_{12} + a_{21} - a_{22}$ and $A_3 = (a_{12} + a_{22})/A_2$.

To show the stability of x^*, let $z = x - x^*$ (i.e. $z(t) = x(t) - x^*$ and $z(t - \tau) = x(t - \tau) - x^*$). Then, we have that

$$\frac{dz(t)}{dt} = \frac{A_1}{A_2}\cdot\frac{z(t - \tau)}{z(t - \tau) + x^* + A_3}(z(t) + x^*)(1 - z(t) - x^*). \tag{7}$$

Notice that the Taylor expansion of Eq. (7)around $z = 0$ can be expressed as

$$\begin{aligned} \frac{dz(t)}{dt} = &- Bz(t-\tau) + \frac{A_1}{A_2}\frac{1 - 2x^*}{x^* + A_3}z(t)z(t-\tau) - \frac{A_1}{A_2}\frac{x^*(1-x^*)}{(x^* + A_3)^2} \\ &z(t - \tau)^2 + \ldots, \end{aligned}$$

where $B = -A_1x^*(1 - x^*)/(A_2(x^* + A))$.

Thus, the linear approximation of Eq. (7) around $z = 0$ can be given by

$$\frac{dz(t)}{dt} \approx -Bz(t - \tau). \tag{8}$$

For this simple time-delay differential equation, its characteristic equation can be easily given by

$$\lambda + Be^{-\lambda\tau} = 0 \tag{9}$$

(Murray, 1989) and if all solutions of this equation have a negative real part (i.e. $\mathrm{Re}\lambda_j < 0$ for all $j = 1, 2, \ldots$), then $z = 0$, corresponding to the interior equilibrium x^* of Eq. (6), is asymptotically stable. Conversely, if there are any solutions of Eq. (9) with $\mathrm{Re}\,\lambda > 0$, then $z = 0$ is unstable.

For the solutions of Eq. (9), it is easy to see that if $B < 0$, then for any $\tau > 0$ Eq. (9) must have a positive real solution. Thus, $z = 0$ must be unstable if $B < 0$. On the other hand, for the situation with $B > 0$, some previous studies (May, 1973; Murray, 1989; Gopalsamy, 1992; etc.) have shown that all possible solutions of Eq. (9) have a negative real part if and only if $B\tau < \pi/2$, and that the real parts of all solutions of Eq. (9) equal exactly zero if $B\tau = \pi/2$. So, $z = 0$ is asymptotically stable if $0 < B < \pi/2\tau$.

From the above analysis, we know that the interior equilibrium of the time-delay imitation dynamics Eq. (6),x^*, is asymptotically stable if and only if

$$\begin{aligned} B &= \tfrac{x^*(1-x^*)(a_{12} - a_{22} + a_{21} - a_{11})}{x^*(a_{11} - a_{12} + a_{21} - a_{22}) + (a_{12} + a_{22})} > 0, \\ \tau &< \tfrac{\pi}{2B} = \tfrac{\pi[x^*(a_{11} - a_{12} + a_{21} - a_{22}) + (a_{12} + a_{22})]}{2x^*(1-x^*)(a_{12} - a_{22} + a_{21} - a_{11})} \\ &= \tfrac{\pi(a_{12}a_{21} - a_{11}a_{22})}{(a_{12} - a_{22})(a_{21} - a_{11})}. \end{aligned} \tag{10}$$

Here, we can also see that if $a_{12} > a_{22}$ and $a_{21} > a_{11}$, then we must have $B > 0$. Thus, the interior equilibrium x^* is stable if not only the payoff matrix satisfies $a_{12} > a_{22}$ and $a_{21} > a_{11}$ but also the time delay τ satisfies $\tau < \pi(a_{12}a_{21} - a_{11}a_{22})/[(a_{12} - a_{22})(a_{21} - a_{11})]$. This result is also exactly similar to Tao and Wang's (1997) results about the stability of evolutionary game dynamics with time delay. For an example with payoff matrix $\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$, the interior equilibrium is $x^* = 1/2$, and the interior equilibrium is stable if $\tau < 5\pi/2$. However, if $\tau > 5\pi/2$, then the dynamics Eq. (6) will display a periodic solution around the interior equilibrium $x^* = 1/2$, and the amplitude of the periodic solution will increase with the increase of τ (see Fig. 1). This implies that the time delay will have a profound impact on the dynamical properties of the imitation system Eq. (6).

2.2. Three-phenotype imitation dynamics with time delay

We now consider a three-phenotype model with three phenotypes denoted by S_i for $i = 1, 2, 3$ and with payoff matrix $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ where we still assume that a_{ij} is positive for all $i, j = 1, 2, 3$. Let x_i denote the frequency of S_i and f_i the expected payoff of S_i with $f_i(\mathbf{x}) = \sum_{j=1}^{3} x_j a_{ij}$ for $i = 1, 2, 3$, where $\mathbf{x} = (x_1, x_2, x_3)$ with $\sum_{i=1}^{3} x_i = 1$. Similar to the definition in the two-phenotype model, when an individual using phenotype

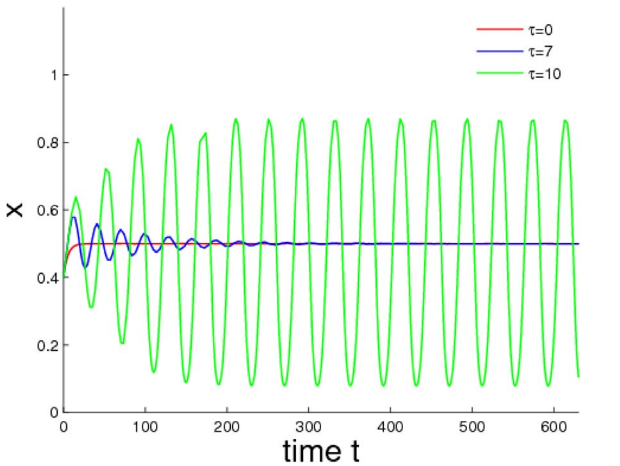

Fig. 1. *Effect of time delay on the two-phenotype imitation dynamics.* In this example, the payoff matrix is taken as $\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$ and the interior equilibrium is $x^* = 1/2$. The numerical analysis of Eq. (6) (with initial $x = 0.4$) also shows clearly that when $\tau < 5\pi/2$, the system state will converge to the equilibrium point $x^* = 1/2$ with damped oscillation (red curve with $\tau = 0$, blue curve with $\tau = 7$); and when $\tau > 5\pi/2$, the system state will converge to a stable periodic solution (green curve with $\tau = 10$). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

S_i plays against an individual using phenotype S_j at time t, the probability (the imitation rate) that the S_j-strategist switches to S_i is defined as

$$F_{ij}(\mathbf{x}(t-\tau)) = \frac{f_i(\mathbf{x}(t-\tau))}{f_i(\mathbf{x}(t-\tau)) + f_j(\mathbf{x}(t-\tau))} \tag{11}$$

for $i, j = 1, 2, 3$. Thus, the time-delay imitation dynamics with three phenotypes can be given by

$$\begin{aligned} \frac{dx_i(t)}{dt} &= x_i(t)\sum_{j\neq i} x_j(t)(F_{ij}(\mathbf{x}(t-\tau)) - F_{ji}(\mathbf{x}(t-\tau))) \\ &= x_i(t)\sum_{j\neq i} x_j(t)\left[\frac{f_i(\mathbf{x}(t-\tau)) - f_j(\mathbf{x}(t-\tau))}{f_i(\mathbf{x}(t-\tau)) + f_j(\mathbf{x}(t-\tau))}\right] \end{aligned} \tag{12}$$

for $i = 1, 2, 3$ (Hofbauer and Sigmund, 1998).

For the time-delay imitation dynamics (12), as an example, only a rock-scissors-paper game with payoff matrix $\begin{pmatrix} c & c-1 & c+s \\ c+s & c & c-1 \\ c-1 & c+s & c \end{pmatrix}$(where $s > -1$ and $c > 1$) is investigated using numerical analysis, where S_1, S_2 and S_3 represent the strategies rock, scissors and paper, respectively (Wesson and Rand, 2016). For this system, it is easy to see that the interior equilibrium is $\mathbf{x}^* = (1/3, 1/3, 1/3)$. We here take $s = 5$ and $c = 2$. The numerical analysis shows that the interior equilibrium $\mathbf{x}^*$ is asymptotically stable when $\tau = 0$ and $\tau = 0.5$ (see Fig. 2). However, when $\tau = 1.5, \tau = 2, \tau = 2.5, \tau = 3$, $\tau = 3.5$ and $\tau = 4$, the system state tends to a stable periodic solution (see Fig. 2). This result strongly implies that

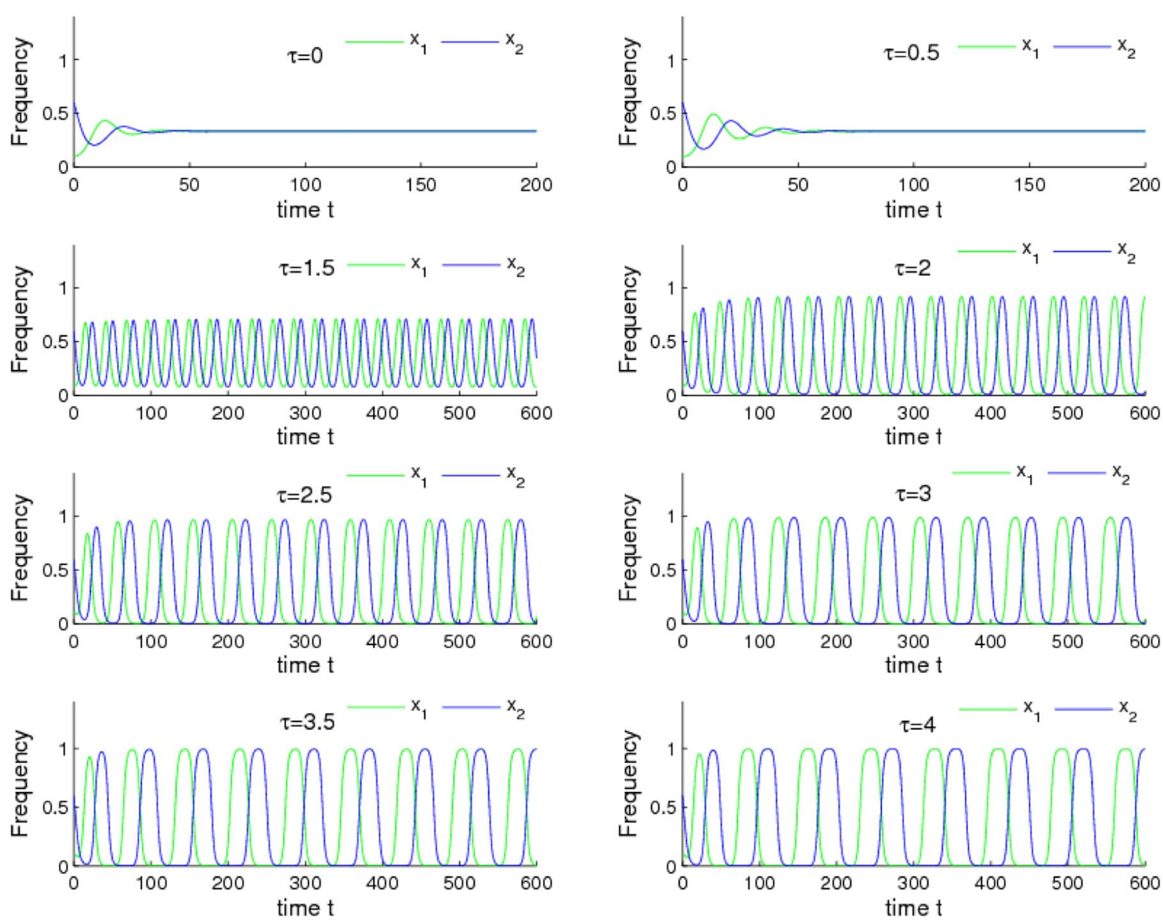

Fig. 2. *Effect of time delay on a three-phenotype imitation dynamics based on the rock- scissors-paper game.* For the payoff matrix $\begin{pmatrix} c & c-1 & c+s \\ c+s & c & c-1 \\ c-1 & c+1 & c \end{pmatrix}$ with $s = 5$ and $c = 2$, and with initial $x_1 = 0.1$ and $x_2 = 0.6$, the interior equilibrium $\mathbf{x}^* = (1/3, 1/3, 1/3)$ is asymptotically stable when $\tau = 0$, or $\tau = 0.5$, and the system state tends a stable periodic solution when $\tau = 1.5, \tau = 2, \tau = 2.5, \tau = 3$, $\tau = 3.5$ and $\tau = 4$.

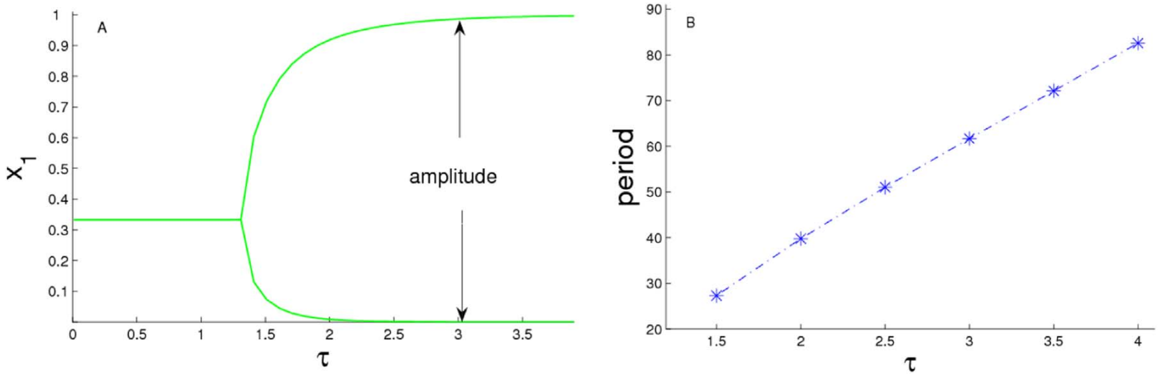

Fig. 3. *Effect of τ on amplitude in a three-phenotype imitation dynamics.* For the dynamics in Fig. 2, (A) when τ is larger than a critical value, the increase of τ will leads to the increase of amplitude of the periodic solution; and (B) the period will also increase with the increase of τ.

there should be a critical value of τ, denoted by $\tilde{\tau}$, such that the interior equilibrium $\mathbf{x}^*$ is stable if $\tau < \tilde{\tau}$. We also see that the system will have a stable periodic solution if $\tau > \tilde{\tau}$, and the amplitude and the period of this stable periodic solution will increase with the increase of τ (Fig. 3).

3. Conclusion

In this study, the imitation dynamics with time delay is investigated, where the probability that an individual with its own strategy i will imitate its opponent's own strategy j is defined as $f_j(t-\tau)/[f_i(t-\tau)+f_j(t-\tau)]$, where $f_k(t-\tau)$ denotes the expected payoff of an individual using strategy k at time $t-\tau$; and, similarly, the probability that an individual with its own strategy i will not imitate its opponent's own strategy j is defined as $f_i(t-\tau)/[f_i(t-\tau)+f_j(t-\tau)]$. For the two-phenotype model, our results show clearly that if the payoff matrix satisfies $a_{12} > a_{22}$ and $a_{21} > a_{11}$, then the interior equilibrium $x^* = (a_{12}-a_{22})/(a_{12}-a_{22}+a_{21}-a_{11})$ must be asymptotically stable if the time delay τ satisfies the inequality $\tau < \pi(a_{12}a_{21}-a_{11}a_{22})/((a_{12}-a_{22})(a_{21}-a_{11}))$, where the term $\pi(a_{12}a_{21}-a_{11}a_{22})/((a_{12}-a_{22})(a_{21}-a_{11}))$ is a critical value of τ. When the time delay is larger than this critical value, the system sate will display a stable periodic solution. This result is exactly similar to a previous study (Tao and Wang, 1997) on the two-phenotype replicator dynamics with time delay. Furthermore, as an example, we investigated a three-phenotype (rock-scissors-paper) imitation dynamics with time delay using numerical analysis, and we found that for the stable periodic solution induced by the time delay and the amplitude will increase with the increase of the time delay. All of our results in this study only provide a small window for revealing the effect of time delay on the imitation dynamics. Our model should be also extended to the stochastic imitation dynamics with time delay in a finite population (Blume, 1993; Szabó and T˝oke, 1998; Wu et al., 2015).

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